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Technical Notes

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Method for Determination of Viscoelastic Parameters Using the Principle of Correspondence

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Nomenclature

E	=	Young's modulus
e_{kl}	=	strain tensor
I	=	area moment of inertia
L	=	length
£	=	Laplace transform
P	=	load
T_{kl}	=	stress tensor
$\bar{T}_{kl}$	=	£$\{T_{kl}\}$
$v(x, 0, t)$	=	tip deflection of beam
$\bar{v}(x, s)$	=	£$\{v\}$
x	=	longitudinal axis of beam
γ_1, γ_2	=	constants appearing in Eq. (8)
δ_{ij}	=	Kronecker symbol, 0 if $i \neq j$, 1 if $i = j$
λ_e, μ_e	=	Lamé constants (elastic)
$\lambda(s), \mu(s)$	=	functions of the Laplace transform parameter s
λ_v, μ_v	=	viscoelastic material constants
τ_1, τ_2	=	relaxation times

Introduction

A PROGRAM is being conducted to study and predict the viscoelastic behavior of composite materials. This study has evolved due to the concern associated with composite ship structures' performance in the event of fire. In this Note, the correspondence principle (CP) is utilized for two types of beams (homogeneous and laminated), to predict the deflections of viscoelastic structures. The main aim is to have viscoelastic solutions that can be compared with the experimental results to determine the viscoelastic parameters.

Eringen[1] has given a lucid account of the CP, also known as the correspondence rule. This principle has been used by Lee[2] for isotropic materials without temperature and by Biot[3] for anisotropic materials. Schapery[4] produced a detailed analysis in which the principle is used to correlate the viscoelastic properties of composites in terms of the constituent properties. For this study, the Kelvin–Voigt model is utilized instead of the Boltzman superposition principle that is used in Ref. 4. The Kelvin–Voigt model has been chosen for its simplicity in that only two viscoelastic parameters need to be determined.

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Palmer et al.[5] conducted a study to develop analytical tools to predict the elevated temperature response of composite structures. The viscoelastic properties were then extrapolated or estimated by experimental tests. Ha and Springer[6] conducted a study to determine the time-dependent viscoelastic stress–strain relationships. Critchfield[7] used Zocher's[8] application of the CP to generate curves that will be compared to the analytical results of this study. Because the material system for this program will be equivalent to the system used by Palmer et al.,[5] the same constants and data are used for validation and comparison purposes.

In comparison to previous studies, the most distinctive feature of this study is to obtain the material constants of viscoelastic and layered viscoelastic materials by simple experiments. For example, if simple cantilevered and sandwich beam experiments are performed, then a plot of the experimental deflection with time can be obtained. By the use of the methodology described, the theoretical plots can also be generated for the same structure. The two plots can then be compared, and the viscoelastic parameters can be extracted by determining the constants that allow the theoretical curve to best fit the real data. At this time, research is under way to devise a simple trial and error method to accomplish this task.

Analysis

In this Note, our purpose is to use the CP to solve simple problems to expose the material constants and to determine their numerical values by comparison with the experimental results. The theorem of correspondence[1] is simply stated as follows: For problems concerning linear viscoelastic solids the solutions with time independent boundaries can be obtained from the solutions of corresponding problems in elasticity. Because of its simplicity, the problem of a homogeneous cantilevered beam was first considered. The second problem of a laminated cantilevered beam was addressed due to its frequent use in composite ship structures. In each problem, the Kelvin–Voigt model of viscoelastic behavior has been used.

Problem 1: Homogeneous Cantilever Beam

The classical solution for a cantilever beam is

$$v(x, 0) = (PL^3/3EI)\left[-\tfrac{1}{2}(1 - x^3/L^3) + \tfrac{3}{2}(1 - x/L)\right]$$

under the condition $\partial v/\partial x = 0$ at $x = 1$ and $y = 0$. All quantities are shown in Fig. 1. In terms of Lamé constants, the modulus E is defined to be

$$E = \frac{\mu_e(3\lambda_e + 2\mu_e)}{\lambda_e + \mu_e} \tag{1}$$

According to the CP, the solution of a cantilever beam formed of a homogeneous viscoelastic material is given by

$$\bar{v}(x, s) = \frac{\bar{P}L^3}{3I}\frac{\lambda(s) + \mu(s)}{\mu(s)[3\lambda(s) + 2\mu(s)]}\left[-\frac{1}{2}\left(1 - \frac{x^3}{L^3}\right) + \frac{3}{2}\left(1 - \frac{x}{L}\right)\right] \tag{2}$$

where $\bar{v}$ is the Laplace transform of $v(x, 0, t)$ for the viscoelastic case, that is,

$$\bar{v} = £[v] = \int_0^\infty e^{-st}v(x, 0, t)\,dt$$

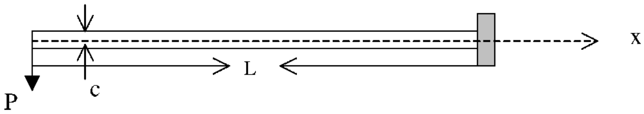

Fig. 1 Homogeneous cantilevered beam.

The form of the functions $\lambda(s)$ and $\mu(s)$ depends on the model to be chosen to describe the viscoelastic behavior. If the Kelvin–Voigt model is chosen, then the constitutive equation is

$$T_{kl} = \left(\lambda_e + \lambda_v \frac{\partial}{\partial t}\right) e_{rr}\delta_{kl} + 2\left(\mu_e + \mu_v \frac{\partial}{\partial t}\right) e_{kl} \tag{3}$$

where T_{kl} is the stress tensor and e_{kl} is the strain tensor. Repeated indices imply summation. Furthermore, λ_v and μ_v are material constants. Taking the Laplace transform of Eq. (3), we obtain

$$\bar{T}_{kl} = (\lambda_e + s\lambda_v)\bar{e}_{rr}\delta_{kl} + 2(\mu_e + s\mu_v)\bar{e}_{kl} \tag{4}$$

When Eq. (4) is compared with the standard model of Hooke's law, the following is obtained:

$$\lambda(s) = \lambda_e + s\lambda_v, \qquad \mu(s) = \mu_e + s\mu_v \tag{5}$$

for a Kelvin–Voigt model. When Eq. (5) is substituted into Eq. (2), the following quantities are introduced:

$$\eta_e = \frac{\lambda_e}{\mu_e}, \qquad \eta_v = \frac{\lambda_v}{\mu_v}, \qquad \tau_1 = \frac{\mu_v}{\mu_e}, \qquad \tau_2 = \frac{3\lambda_v + 2\mu_v}{3\lambda_e + 2\mu_e}$$

$$r = \frac{\tau_2}{\tau_1}, \qquad z = \frac{1+\eta_v}{1+\eta_e} = \frac{[1 + r(2+3\eta_e)]}{3+3\eta_e} \tag{6}$$

where τ_1 and τ_2 have dimension of time. With the quantities as stated in Eqs. (5) and (6) and substituted into Eq. (2), the inverse of the transform is obtained. The load function $P(t)$ is described hereafter for two cases: Case 1 is where

$$P(t) = P = \text{const}$$

Thus, $\bar{P}(s) = P/s$. Case 2 is where

$$P(t) = \begin{cases} P = \text{const}, & 0 \le t < t_m \\ 0, & t > t_m \end{cases}$$

Thus, $\bar{P}(s) = P(1 - e^{-st_m})/s$.

For case 1 the inverse transform is

$$v(x,t) = kPF_1(x,t), \qquad k = L^3/3I \tag{7a}$$

where

$$F_1(x,t) = [1/(1-r)][(1-r) + re^{-t/\tau_2} - e^{-t/\tau_1} + z(e^{-t/\tau_1} - e^{-t/\tau_2})]\left[-\tfrac{1}{2}(1 - x^3/L^3) + \tfrac{3}{2}(1 - x/L)\right] \tag{7b}$$

For case 2 the solution (7a) is valid for t in the range $0 < t < t_m$, whereas for $t > t_m$,

$$v(x,t) = kPF_2(x,t) \tag{7c}$$

$$F_2(x,t) = [1/(1-r)]\left[re^{-t/\tau_2} - e^{-t/\tau_1} + z(e^{-t/\tau_1} - e^{-t/\tau_2}) - (z-1)e^{(t_m - t)/\tau_1} + (z-r)e^{(t_m-t)/\tau_2}\right]\left[-\tfrac{1}{2}(1 - x^3/L^3) + \tfrac{3}{2}(1 - x/L)\right] \tag{7d}$$

From Eq. (7), it is observed that for a Kelvin–Voigt viscoelastic model there are two unknown parameters, τ_1 and τ_2.

Now that the purely viscoelastic part of the solution has been obtained, note that Eq. (7a) vanishes at $t = 0$, as it should. There may already be a displacement in existence that is not equal to zero. To include this part of the solution, it is recalled that for displacements in linear elastic solids, which are governed by linear partial differential equations, that is, the Navier equations (for example, see Eringen[1]), a constant multiple of Eq. (7) added to a constant is also a solution. Thus, a linear combination of Eqs. (7a) and (7b) is

$$v(x,t) = kP[1/E + \gamma_1 F_1(x,t) + \gamma_2 F_2(x,t)] \qquad \text{where} \qquad k = L^3/3I \tag{8}$$

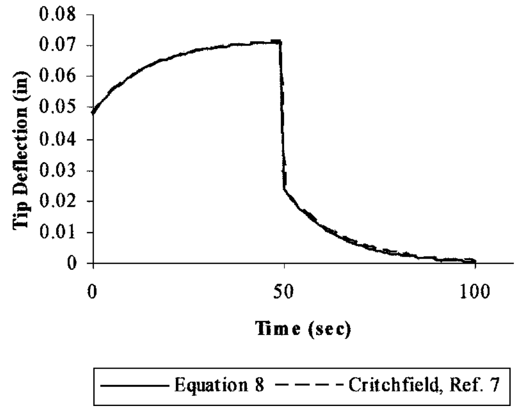

Fig. 2 Deflection history for the homogeneous cantilevered beam.

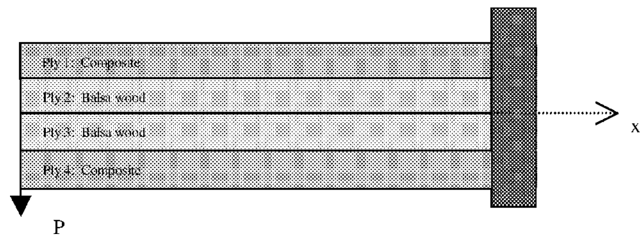

Fig. 3 Laminated composite beam.

Because of the introduction of the parameters γ_1 and γ_2, the complete solution depends on five parameters: τ_1, τ_2, z, γ_1, and γ_2. If $\gamma_2 = 0$, then the solution of Eq. (8) is applicable for $P = \text{const}$.

As a verification of solution (8), the results of Critchfield[7] have been taken as the values of the following parameters: $P = 10$ lb (44.5 N), $L = 60$ in. (1.52 m), $w = 6$ in. (0.152 m), $2c = \text{height} = 1$ in. (0.0254 m), $E = 30E6$ psi (206.8E9 Pa), $\nu = 0.3$, $\tau_1 = 15$ s, $\tau_2 = 10.5$ s, and $r = 0.7$. The values of η_e and z by using $\nu = 0.3$ are, respectively,

$$\eta_e = \frac{2\nu}{1-2\nu}, \qquad z = \frac{1+\eta_v}{1+\eta_e} = \frac{[(2+3\eta_e)r + 1]}{3+3\eta_e}$$

As discussed in the Introduction, the viscoelastic parameters can be found by fitting the analytical data to the experimental data. When $\gamma_1 = 1.64E{-}8$ psi ($0.11E{-}3$ Pa) and $\gamma_2 = 17E{-}9$ psi ($0.12E{-}3$ Pa) are chosen, the analytical curve generated from Eq. (8) compares very favorably to the experimental curve obtained from Ref. 7, as shown in Fig. 2.

Problem 2: Laminated Cantilevered Beam

Figure 3 shows a laminated cantilevered beam composed of a total of four plies. For this initial study, the transverse shear effects are not considered. It is acknowledged that shear effects play a significant role in composite sandwich beams, especially those containing low shear stiffness cores. Because the demonstration of the basic use of the CP is the main focus of this study, the shear effects are ignored. Future work in which the analytical curve must be fitted to the actual experimental data will include the transverse shear effects.

As has been described by Gibson,[9] for four laminated symmetrically oriented plates, the effective modulus E_f for $N = 4$ is given by

$$E_f = \tfrac{1}{8}[E_1 + 7E_2]$$

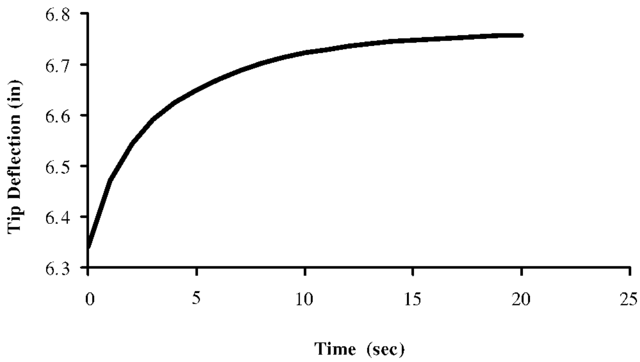

Fig. 4 Deflection history for laminated cantilevered beam.

which, in terms of Lamé constants, is

$$E_f = \frac{1}{8}\left[\frac{\mu_1(3\lambda_1 + 2\mu_1)}{\lambda_1 + \mu_1} + \frac{7\mu_2(3\lambda_2 + 2\mu_2)}{\lambda_2 + \mu_2}\right]$$

When the preceding procedure is followed and the CP used, the deflection at $x = 0$ is given as

$$v(0, t) = \frac{PL^3}{3I}\left[\frac{1}{E_f} + \frac{8\gamma_1}{d}\sum_0^4 \frac{g(\alpha_k)}{p'(\alpha_k)} e^{\alpha_k t}\right]$$

where d is a constant and $g(s)$ and $p(s)$ are polynomials. The constants α_k are the roots of $p(s) = 0$ with $\alpha_1 = 0$. For a sample calculation (using properties obtained from Ref. 5, with the subscript 1 representing the composite properties and 2 being the balsa wood properties), the solution is shown in Fig. 4: $E_1 = 4E6$ psi ($27.6E9$ Pa), $E_2 = 10E3$ psi ($68.9E6$ Pa), $\nu_1 = 0.14$, $\nu_2 = 0.011$, $\tau_{11} = 5$ s, $\tau_{12} = 2.5$ s, $\tau_{21} = 1.0$ s, $\tau_{22} = 0.5$ s, and $\gamma_1 = 1.667E{-8}$ psi ($0.11E{-3}$ Pa).

Conclusions

A study has been undertaken to determine the viscoelastic parameters by using the CP. The CP was applied to the elastic solution for the deflection of a cantilevered beam to obtain the resulting viscoelastic equation. The Kelvin–Voigt model was then used to express Lamé constants in terms of the viscoelastic parameters. The equation was Laplace transformed to express the deflection equation in terms of length and time. The resulting equation was then plotted and adjusted to fit curves obtained from previous studies.[5,7] From the curve fitting, the viscoelastic parameters were extracted.

Further work in this area will produce experimental results that will be used to determine the viscoelastic constants, and these will then be confirmed by the use of the CP. The main attraction of this method is that the extracted viscoelastic constants are determined only once and then can be used in the analysis of more complicated structures as long as the same material system is used.

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Curvature Corrections for Algebraic Reynolds Stress Modeling: A Discussion

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I. Introduction

TURBULENT flows are known to be sensitive to streamline curvature. It is also clear that linear eddy-viscosity turbulence models (EVMs) are completely insensitive to the effects of streamline curvature. On the other hand, differential Reynolds stress models (RSMs) are able to capture these effects. However, RSMs have not become a standard tool in practical fluid-flow simulations, owing to their complexity, large computational work load, and typically unfavorable effect on numerical stability. Therefore, explicit algebraic Reynolds stress models (EARSMs) have become increasingly popular during recent years. EARSMs are two-equation models sharing much of the computational manageability of the EVMs while partially retaining the more realistic physical background of the underlying RSM. However, the sensitivity to the streamline curvature is partially lost through the weak-equilibrium assumption invoked to derive algebraic RSMs. It has been shown by several authors that, in principle, this deficiency can be alleviated by assuming the weak equilibrium in a suitable curvilinear stream-following coordinate system.

The algebraic RSM (ARSM) formulation for curved flows will be revisited. This is to show how the weak equilibrium assumption can be invoked in a suitable curvilinear coordinate system to minimize approximately the resulting error for curved flows. Such approximations are proposed in the literature based on the rotation rate of 1) the velocity vector,[1] 2) the acceleration vector,[2,3] and 3) the principal system of the strain-rate tensor.[3–6] The first approach is, in principle, not fully generalizable because of its lack of Galilean invariance.

The purpose of this Note is to show that the acceleration method is also generally invalid. The observed singular behavior of the acceleration method is discussed in general, and its failure is numerically demonstrated in a plane duct flow including a 180-deg bend with a small radius of curvature.[7,8] The behavior of the acceleration method will be compared with that of the strain-rate and velocity-based methods.

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